Minimum-Time Loop Maneuvers of Jet Aircraft

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Minimum-time loop maneuvers of jet aircraft are investigated by means of the calculus of variations. The optimal control (lift coefficient and thrust) is determined as a function of the state variables and Lagrange multipliers for an arbitrary maneuver in the vertical plane. Intermediate thrust arcs and intermediate lift arcs with minimum thrust are shown to be nonoptimal for the dynamic models considered. Rules are established for joining trajectory segments with different types of control behavior. Using these results, optimal loop trajectories are generated for a given set of initial conditions and a large number of values of maximum lift coefficient and maximum thrust. The control history and the maneuver time are found to be strongly affected by the latter two parameters. Two aerodynamic models are considered, one relatively simple and the other somewhat more complicated.

Nomenclature

a	= speed of sound
C_D	= drag coefficient = $C_{D0}+KC_L^2$
C_{D0}	= zero-lift drag coefficient
C_L	= lift coefficient
D	= drag = qSC_D
g	= acceleration of gravity
H	= variational Hamiltonian
J	= performance index to be optimized
K	= airplane efficiency factor
L	= lift = qSC_L
M	= Mach number = V/a
n	= load factor (normal acceleration in g's)
p	= atmospheric pressure
p_0	= atmospheric pressure at reference altitude
q	= dynamic pressure = $\rho V^2/2=\kappa p M^2/2$
S	= wing area of aircraft
Sw	= reciprocal of dimensionless wing loading = $\kappa p_0 S/2W$
t	= time
T	= thrust
Tw	= thrust to weight ratio = T/W
V	= velocity magnitude
W	= weight of aircraft
x	= horizontal range
y	= altitude
α	= angle of attack
γ	= flight path angle
η	= dimensionless altitude = gy/a^2
κ	= ratio of specific heats of air = 1.4
λ_j	= jth Lagrange multiplier for differential constraints
μ_j	= jth Lagrange multiplier for inequality constraints
ξ	= dimensionless range = gx/a^2
ρ	= atmospheric density
τ	= dimensionless time = gt/a

Superscripts and Subscripts

$(\dot{\ })$	= derivative with respect to time
$(\ )'$	= derivative with respect to dimensionless time
$(\ )_c$	= value at a corner point
$(\ )_f$	= final value
$(\ )_i$	= initial value
$(\ )_{int}$	= intermediate value
$(\ )_{max}$	= maximum value
$(\ )_{min}$	= minimum value

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Introduction

In recent years, optimal control theory has been applied to various performance optimization problems associated with high-speed aircraft. These problems include minimum-time and minimum-fuel climbs and dives,[1-11] minimum-time and minimum-fuel turns,[11-21] and range maximization and fuel minimization in cruise.[5,22-29] Relatively little investigation of maneuvers associated with a large flight-path angle has been made, however.

In this paper, minimum-time loop maneuvers (maneuvers in the vertical plane in which the flight-path angle increases monotonically from 0 to 360 deg) are studied. These maneuvers are interesting from a theoretical point of view, but more importantly, their investigation is motivated by a desire to make system tradeoffs between maneuver requirements, maximum-lift buffet limits, and thrust/weight ratios. Optimal lift and thrust histories for loop maneuvers are obtained, and the effects of maximum lift coefficient and maximum thrust on the control history and maneuver time are investigated. References 30 and 31 have discussed the climbing part of loop maneuvers.

Analytical Formulation

Several approximations may be used to analyze the motion of an aircraft in the vertical plane.[5] The quasisteady approximation (as defined in Ref. 5), in which accelerations are neglected completely, is adequate for treating the climb performance of subsonic aircraft. However, this approximation is inadequate for a loop maneuver, which requires changes in flight-path angle of 360 deg. The energy-state approximation, in which energy is the only state variable, is quite useful for analyzing the minimum-time and maximum-range performance of high-speed aircraft, provided that the acceleration normal to the flight path can be neglected and that the flight is nearly horizontal. In a loop maneuver, neither of these assumptions is justified. A somewhat more accurate approximation involves considering the velocity and altitude as state variables and the flight-path angle as a control variable. This approach is also not suitable for analysis of a loop maneuver, because acceleration normal to the flight path is again neglected. Since accelerations parallel to and normal to the flight path are much more important in a loop maneuver than in a climb, the selection of velocity, flight-path angle, and altitude as state variables, and lift coefficient, thrust, and zero-lift drag coefficient as control variables is appropriate for analysis of a loop maneuver.

If the aircraft is modeled as a point mass, the equations of motion are[32]

$$(W/g)\,\dot{V} = T\cos\alpha - D - W\sin\gamma$$

$$(W/g)\,V\dot{\gamma} = T\sin\alpha + L - W\cos\gamma$$

$$\dot{x} = V\cos\gamma$$

$$\dot{y} = V\sin\gamma$$

In order to minimize the complexity of the problem, it is assumed that the atmospheric pressure and temperature are constant during the maneuver, the weight of the aircraft is constant, the angle of attack is sufficiently small so that the component of thrust normal to the flight path is much smaller than the lift ($T\alpha \ll L$), the normal acceleration and Mach number are unconstrained, the zero-lift drag coefficient is constant, and the upper and lower bounds on lift coefficient and thrust are constant. If the lift coefficient and thrust-to-weight ratio are chosen to be control variables and the equations of motion are expressed in dimensionless form, the state equations and control constraints are expressible as

$$M' = Tw - SwM^2(C_{D0} + KC_L^2) - \sin\gamma \tag{1}$$

$$\gamma' = (SwM^2C_L - \cos\gamma)/M \tag{2}$$

$$\xi' = M\cos\gamma \tag{3}$$

$$\eta' = M\sin\gamma \tag{4}$$

$$(C_L - C_{L_{max}})(C_L - C_{L_{min}}) \le 0 \tag{5}$$

$$(Tw - Tw_{max})(Tw - Tw_{min}) \le 0 \tag{6}$$

Equation (1) explicitly assumes the drag coefficient C_D to be a parabolic function of C_L.

Analysis by the Calculus of Variations

Minimum-time loop trajectories may be determined by solving a Mayer problem of variational calculus with performance index

$$J = \tau_f - \tau_i \tag{7}$$

and variational Hamiltonian

$$H = \lambda_1[Tw - Sw\,M^2(C_{D0} + KC_L^2) - \sin\gamma] + (\lambda_2/M)(SwM^2C_L - \cos\gamma) + \lambda_3 M\cos\gamma + \lambda_4 M\sin\gamma \tag{8}$$

The Euler-Lagrange equations are

$$\lambda_1' = \lambda_1 2SwMC_D - (\lambda_2/M^2)(SwM^2C_L + \cos\gamma) - \lambda_3\cos\gamma - \lambda_4\sin\gamma \tag{9}$$

$$\lambda_2' = \lambda_1\cos\gamma - (\lambda_2/M)\sin\gamma + \lambda_3 M\sin\gamma - \lambda_4 M\cos\gamma \tag{10}$$

$$\lambda_3' = 0 \rightarrow \lambda_3 = \text{constant} \tag{11}$$

$$\lambda_4' = 0 \rightarrow \lambda_4 = \text{constant} \tag{12}$$

The Hamiltonian and each of the Lagrange multipliers must be continuous at all corners (points of discontinuity in the control), according to the corner conditions of variational calculus, since there are no interior point constraints or state variable inequality constraints. Moreover,

$$H \equiv -1 \tag{13}$$

since H is not explicitly time-dependent, and final time is to be minimized.

Characteristics of Optimal Subarcs

The control variables C_L and Tw are not coupled in the Hamiltonian. Hence, H may be optimized with respect to each control variable independently, using the minimum principle.[33] Since H is linear in Tw, H is minimized with respect to Tw when

$$Tw = \begin{cases} Tw_{max} & \text{if } \lambda_1 < 0 \\ Tw_{int}\ (Tw_{min} < Tw < Tw_{max}) & \text{if } \lambda_1 = 0 \text{ (for a finite time interval)} \\ Tw_{min} & \text{if } \lambda_1 > 0 \end{cases}$$

An intermediate thrust arc is a singular arc, since $\partial H/\partial Tw = \lambda_1 = 0$ and $\partial^2 H/\partial Tw^2 = 0$ on this arc. Additional tests are needed to determine whether such an arc can be optimal. One necessary condition for optimality is that

$$0 = \frac{d}{d\tau}\left(\frac{\partial H}{\partial Tw}\right) = \lambda_1' = \frac{1}{M^2}(M - 2\lambda_2\cos\gamma) \tag{14}$$

where use has been made of Eqs. (9) and (13) in obtaining the last equality.

The generalized Legendre-Clebsch condition[33] requires that

$$\frac{\partial}{\partial Tw}\frac{d^2}{d\tau^2}\left(\frac{\partial H}{\partial Tw}\right) = \frac{4\lambda_2\cos\gamma - M}{M^3} \le 0 \tag{15}$$

In other words, $1/M^2 \le 0$ from Eqs. (14) and (15). Since M must be real, the generalized Legendre-Clebsch condition is not satisfied. Hence, intermediate thrust arcs are not optimal.

H is stationary with respect to C_L when

$$\partial H/\partial C_L = SwM(-2\lambda_1 KC_L M + \lambda_2) = 0 \tag{16}$$

that is, when either

$$\lambda_1 = \lambda_2 = 0 \quad \text{or} \quad C_L = \lambda_2/2KM\lambda_1 \tag{17}$$

Only the latter possibility need be considered, since the former implies a nonoptimal singular arc.

Since H is quadratic in C_L, the preceding choice of C_L produces a local minimum in H if $\lambda_1 < 0$ and a local maximum if $\lambda_1 > 0$. Hence, if $\lambda_1 < 0$ ($Tw = Tw_{max}$), the optimal value of C_L is either $C_{L_{max}}$, $C_{L_{min}}$, or an intermediate value determined from Eq. (17). If $\lambda_1 > 0$ ($T_w = Tw_{min}$), however, the optimal value of C_L can only be $C_{L_{max}}$ or $C_{L_{min}}$. The optimal value of C_L as a function of λ_1, λ_2, and M can be determined by applying the minimum principle. The results are summarized in Table 1. In Table 1, the quantity $C_{L_{avg}}$ is defined to be

$$C_{L_{avg}} = (C_{L_{max}} + C_{L_{min}})/2$$

There are thus five qualitatively different control combinations, or five different trajectory subarcs, which may be used to construct an optimal trajectory.

Sequences of Subarcs

There are certain restrictions on which subarcs can be joined together in constructing an optimal trajectory. This may be seen by first establishing three conditions that must

Table 1 Optimal control as a function of the Lagrange multipliers

Subarc number	λ_1	λ_2	Tw	C_L
II	<0	$2KC_{L_{max}}M\lambda_1<\lambda_2<2KC_{L_{min}}M\lambda_1$	Tw_{max}	$C_{L_{int}}=\lambda_2/2KM\lambda_1$
I	<0	$\leq 2KC_{L_{max}}M\lambda_1$	Tw_{max}	$C_{L_{max}}$
III	<0	$\geq 2KC_{L_{min}}M\lambda_1$	Tw_{max}	$C_{L_{min}}$
IV	>0	$<2KC_{L_{avg}}M\lambda_1$	Tw_{min}	$C_{L_{max}}$
V	>0	$>2KC_{L_{avg}}M\lambda_1$	Tw_{min}	$C_{L_{min}}$

hold at a corner, or junction of two subarcs:

1) The state variables, Lagrange multipliers, and the Hamiltonian must be continuous at a corner.[33]

2) Tw can be discontinuous only if $(\lambda_1)_c=0$, and C_L can be discontinuous only if either

$$(\lambda_1)_c=(\lambda_2)_c=0$$

or

$$(\lambda_2)_c/2KM(\lambda_1)_c=C_{L_{avg}} \quad \text{and} \quad (\lambda_1)_c>0$$

This condition follows from the fact that each control variable is continuous at a corner if H is regular in that variable (i.e., if H has a unique minimum in that variable, subject to the stated constraints).[34]

3) If $\lambda_1=\lambda_2=0$ at any point on an optimal trajectory, λ_1 must change from negative to positive at that point. This follows from the fact that, if $\lambda_1=\lambda_2=0$ at some point, Eqs. (9) and (13) and continuity of λ_3 and λ_4 lead to the conclusions that λ_1' is continuous and $\lambda_1'=1/M>0$ at that point.

From these conditions, it can be shown that the transitions I→III, III→I, IV→II, IV→III, V→II, and V→I are impossible, and that the transitions II→IV, III→IV, II→V and I→V can occur only if both λ_1 and λ_2 vanish simultaneously. The latter four transitions involve a simultaneous change in both control variables and hence are relatively unlikely. Moreover, if $\lambda_3=\lambda_4\equiv 0$ (the case in which final range and altitude are not specified), these transitions are impossible because Eq. (13) is violated. Thus, only the five transitions I→II, III→II, I→IV, III→V, and IV→V and their inverses are of real practical interest.

Since no terminal constraints on the state have been imposed in the foregoing analysis, any time-optimal maneuver in the vertical plane, not just loop maneuvers, will be composed of these subarcs. Loop maneuvers are not likely to include subarcs with minimum lift coefficient, since the flight-path angle increases monotonically. Hence, only subarcs I, II, and IV are of real interest from now on.

In previous work considering minimum-time climb performance, the control variable was taken to be V, γ, or α (equivalent to C_L), with the thrust held at its maximum level. Optimal solutions were found to have intermediate lift and maximum thrust.[1-5,31] In target chasing maneuvers,[30] the control was taken to be lift coefficient or normal force while using maximum thrust. Minimum-time solutions consist of paths with maximum thrust and maximum normal acceleration or maximum lift coefficient.

Numerical Solution of the Two-Point Boundary-Value Problem

In order to determine minimum-time loop trajectories, a two-point boundary-value problem must be solved. In general, extremal trajectories may be generated by guessing initial values for the Lagrange multipliers and integrating the state equations and Euler-Lagrange equations forward, subject to the specified initial conditions on the state variables. The optimal control can be determined in terms of the state and Lagrange multipliers using the minimum principle. In this problem, if the parameters λ_3, λ_4, and $\lambda_1(\tau_i)$ are guessed, differential equations for M, γ, ξ, η, and λ_1 can be integrated forward. The condition $H\equiv -1$ may be used to determine $\lambda_2(\tau)$. $C_L(\tau)$ and $Tw(\tau)$ can be determined from λ_1, λ_2, and M according to Table 1. The final time τ_f can be chosen to be the time at which $\gamma(\tau)=2\pi$. A three-dimensional search must be made for the set of parameters $(\lambda_1)_i$, λ_3, and λ_4, which cause the following terminal conditions to be satisfied:

M_f	specified	or	$(\lambda_1)_f=0$
ξ_f	specified	or	$\lambda_3=0$
η_f	specified	or	$\lambda_4=0$

Data describing the aircraft and atmospheric conditions are given in Table 2. An initial speed of Mach 0.9 and an initial altitude of 20,000 ft were assumed.

Table 2 Aircraft and atmospheric data

Item	Data
Aircraft	$W=18{,}000$ lb $S=220$ ft^2 $Tw_{min}=0$ For simplified model: $C_{D_0}=0.02$ $K=0.2$ For more realistic model: $C_{D_0}=\begin{cases}0.02, M<0.93\\ 0.02+0.2(M-0.93), 0.93\leq M<1.03\\ 0.04+0.06(M-1.03), 1.03\leq M<1.10\\ 0.0442-0.007(M-1.10), 1.10\leq M\end{cases}$ $K=\begin{cases}0.2, M<1.15\\ 0.2+0.246(M-1.15), 1.15\leq M\end{cases}$ $Tw_{max}=0.0405(1+0.597297M^2)Swe^{-1.4\eta}$ (0.5 at altitude of 20,000 ft and speed of Mach 0.9)
Atmosphere	Reference altitude: 20,000 ft $p=972.49$ psf $a=1037.26$ ft/s $\kappa=1.4$
Gravity	$g=32.1741$ ft/s^2

Numerical Results for the Case M_f, ξ_f, and η_f Unspecified

If the final Mach number, range, and altitude are unspecified, only a one-dimensional search [for the value of

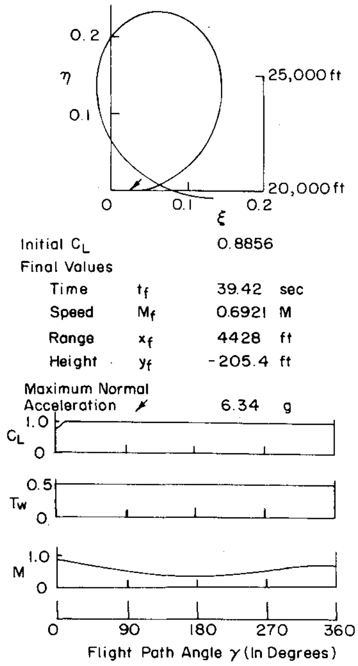

Fig. 1 Control and state histories: $C_{L_{max}} = 1.0$, $Tw_{max} = 0.5$ (region A).

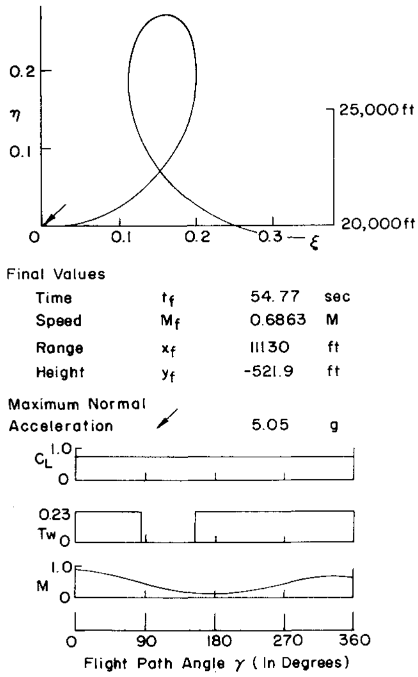

Fig. 3 Control and state histories: $C_{L_{max}} = 0.75$, $Tw_{max} = 0.23$ (region E).

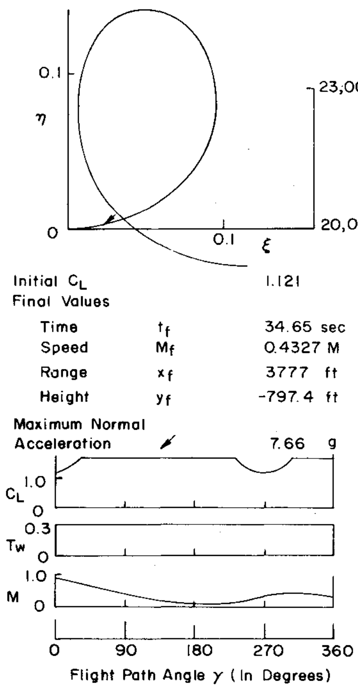

Fig. 2 Control and state histories: $C_{L_{max}} = 1.6$, $Tw_{max} = 0.3$ (region C).

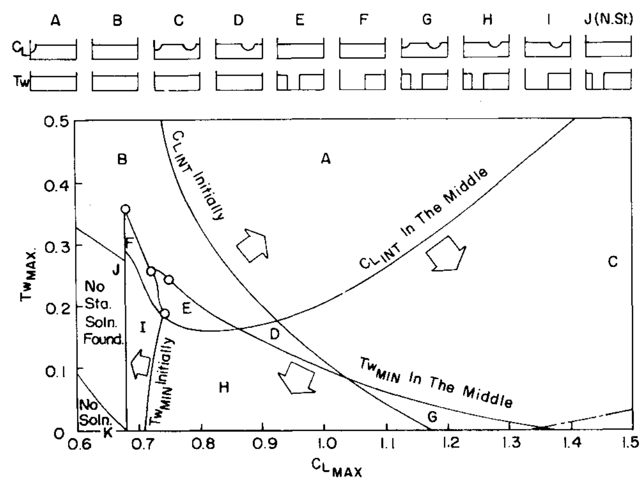

Fig. 4 Control history map (M_f, ξ_f, η_f free).

$(\lambda_l)_i$ such that $(\lambda_l)_f = 0]$ need be made, since λ_3 and λ_4 are identically zero. In this special case, the four-state variable optimal control problem reduces to a two-state variable optimal control problem. Optimal paths are shown in Figs. 1-3 for three sets of values of $C_{L_{max}}$ and Tw_{max}. Figure 4 was constructed by determining optimal trajectories for a large number of values of $C_{L_{max}}$ and Tw_{max} for the fixed set of initial conditions just described. The portion of the $Tw_{max} - C_{L_{max}}$ plane which is of practical significance can be divided into a number of regions according to the control behavior along a minimum-time loop trajectory. The boundary line labeled "$C_{L_{int}}$ Initially" indicates that the C_L program begins with a $C_{L_{int}}$ arc in the region to the right of that line. The label "$C_{L_{int}}$ In the Middle" indicates that the C_L program has a $C_{L_{int}}$ arc in its middle part in the region of the plane below that line. The lines labeled "Tw_{min}..." have functions similar to those of the lines labeled "$C_{L_{int}}$...."

The regions determined by these boundary lines are denoted by the letters A to K, corresponding to the optimal C_L and Tw programs shown in the upper part of the figure, except for regions J and K. For region J, no stationary solutions have been found yet, although the maneuver can be completed. The program shown is the admissible solution that gives the shortest maneuver time. For region K, no control programs are plotted, since the maneuver cannot be completed.§ The flight-path angle cannot be increased beyond 90 deg because of insufficient lift at low speeds. Regions C, D, and G have a second stationary solution, without the $C_{L_{int}}$ arc in the middle of the path. However, these paths produce larger maneuver times than those shown in Fig. 4.

The boundaries separating regions B, E, and F between the small circles have additional interesting characteristics. There are three locally optimal solutions of types B, E, and F, which yield different values of maneuver time in this portion of the $Tw_{max} - C_{L_{max}}$ plane. In region B, the time for solution type B is shortest among the three possibilities, and, in region F, the time for solution type F is shortest. The minimum-time

§The numerical analysis treated only trajectories with $\dot{\gamma} \geq 0$. The possibility of constructing proper solutions for regions J and K by starting the maneuver with a dive was not investigated.

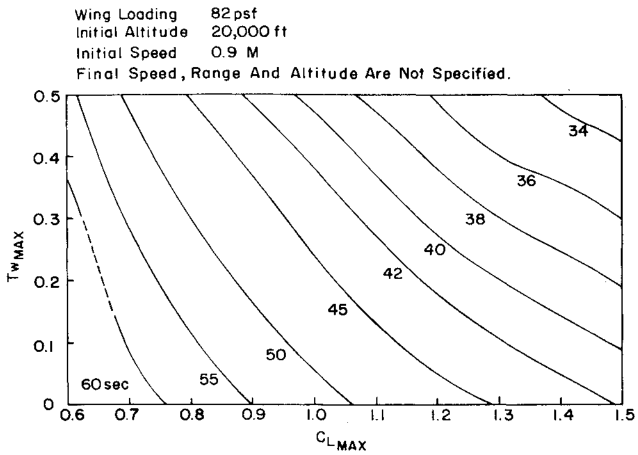

Fig. 5 Contours of time for maneuver (M_f, ξ_f, η_f free).

solution switches from one type of solution to the other across the boundary line between regions B and F. Similar solution switchings have been observed across the boundary lines between regions B and E and regions E and F.

Contours of minimum maneuver time are plotted in the $C_{L_{max}} - Tw_{max}$ plane in Fig. 5. Increasing Tw_{max} and $C_{L_{max}}$ tends to produce shorter maneuver times. The dashed portion of the 60-s contour is in region J.

From the viewpoint of inlet, duct, and engine design, the control histories of lift and thrust shown in Figs. 1-3 present a severe design requirement. Maximum thrust at a high angle of attack and low speed is required for a minimum-time maneuver. (Note that this conclusion does not contradict the assumption $T\alpha \ll L$ made in formulating the problem. For high-speed aircraft, buffet limitations generally permit neglecting $T\alpha$ relative to lift even though α is not really very small.) This will produce distortion in the inlet and duct airflow and may induce compressor stall of the engine. In addition, the transient effect of abrupt thrust level changes on the inlet and duct airflow and on the engine performance might be quite significant, especially at low speeds. Numerical results for cases in which ξ_f or ξ_f and η_f are specified are presented in Ref. 35.

Analysis of a More Realistic Dynamic Model

In order to obtain the features of a realistic minimum-time loop maneuver, some of the simplifying assumptions made previously will now be removed. It is now assumed that the atmospheric pressure varies with altitude according to $p = p_0 e^{-\kappa\eta}$, C_{D0} and κ are functions of Mach number, and Tw_{max} and Tw_{min} are functions of Mach number and altitude. Constraints on normal acceleration due to aircraft structural limits or the pilot's physiological limits are also considered.

Generally, the normal acceleration constraint is effective only when the dynamic pressure ($q = ½\kappa p M^2$) is relatively large. The lift coefficient is then bounded above and below by

$$C_{L_X} \doteq n_{max} / Sw\, e^{-\kappa\eta} M^2$$

and

$$C_{L_N} = n_{min} / Sw\, e^{-\kappa\eta} M^2$$

respectively. Thus the normal acceleration constraints may be handled as constraints on C_L which depend explicitly on Mach number and altitude. In the relatively low dynamic pressure range, the constant upper and lower bounds on C_L are more restrictive than the state-dependent bounds imposed by the normal acceleration limitation. In the high dynamic pressure range, the reverse is true.

Necessary conditions of optimality for minimum-time maneuvers with this dynamic model are

$$M' = Tw - Sw\, e^{-\kappa\eta} M^2 (C_{D0} + KC_L^2) - \sin\gamma \tag{18}$$

$$\gamma' = (1/M)(Sw e^{-\kappa\eta} M^2 C_L - \cos\gamma) \tag{19}$$

$$\xi' = M \cos\gamma \tag{20}$$

$$\eta' = M \sin\gamma \tag{21}$$

$$\begin{aligned}\lambda_1' =\ & \lambda_1 Sw\, e^{-\kappa\eta} M\left[2C_D + M\left(\frac{\partial C_{D0}}{\partial M} + \frac{\partial K}{\partial M} C_L^2\right)\right] \\ & - \frac{\lambda_2}{M^2}(Sw\, e^{-\kappa\eta} M^2 C_L + \cos\gamma) - \lambda_3 \cos\gamma - \lambda_4 \sin\gamma \\ & + \mu_1 \left[\left(\frac{\partial C_{L_{max}}}{\partial M} + \frac{\partial C_{L_{min}}}{\partial M}\right) C_L - \left(\frac{\partial C_{L_{max}}}{\partial M} C_{L_{min}} \right.\right. \\ & \left.\left. + \frac{\partial C_{L_{min}}}{\partial M} C_{L_{max}}\right)\right] + \mu_2 \left[\left(\frac{\partial Tw_{max}}{\partial M} + \frac{\partial Tw_{min}}{\partial M}\right) Tw \right. \\ & \left. - \left(\frac{\partial Tw_{max}}{\partial M} Tw_{min} + \frac{\partial Tw_{min}}{\partial M} Tw_{max}\right)\right]\end{aligned} \tag{22}$$

$$\lambda_2' = \lambda_1 \cos\gamma - (\lambda_2/M)\sin\gamma + \lambda_3 M \sin\gamma - \lambda_4 M \cos\gamma \tag{23}$$

$$\lambda_3 = \text{constant} \tag{24}$$

$$\begin{aligned}\lambda_4' =\ & -\lambda_1 Sw\, e^{-\kappa\eta} M^2 C_D + \lambda_2 Sw\kappa\, e^{-\kappa\eta} M C_L \\ & + \mu_1 \left[\left(\frac{\partial C_{L_{max}}}{\partial \eta} + \frac{\partial C_{L_{min}}}{\partial \eta}\right) C_L - \left(\frac{\partial C_{L_{max}}}{\partial \eta} C_{L_{min}} \right.\right. \\ & \left.\left. + \frac{\partial C_{L_{min}}}{\partial \eta} C_{L_{max}}\right)\right] + \mu_2 \left[\left(\frac{\partial Tw_{max}}{\partial \eta} + \frac{\partial Tw_{min}}{\partial \eta}\right) Tw \right. \\ & \left. - \left(\frac{\partial Tw_{max}}{\partial \eta} Tw_{min} + \frac{\partial Tw_{min}}{\partial \eta} Tw_{max}\right)\right]\end{aligned} \tag{25}$$

$$(C_L - C_{L_{max}})(C_L - C_{L_{min}}) \leq 0 \tag{26}$$

$$(Tw - Tw_{max})(Tw - Tw_{min}) \leq 0 \tag{27}$$

$$\begin{aligned}0 = & -\lambda_1 Sw\, e^{-\kappa\eta} M^2 2KC_L + \lambda_2 Sw\, e^{-\kappa\eta} M - \mu_1 \\ & \times [(C_{L_{max}} + C_{L_{min}}) - 2C_L]\end{aligned} \tag{28}$$

$$0 = \lambda_1 - \mu_2 [(Tw_{max} + Tw_{min}) - 2Tw] \tag{29}$$

$$-\lambda_1 KSw\, e^{-\kappa\eta} M^2 + \mu_1 \geq 0 \tag{30}$$

$$\mu_1 \geq 0 \quad \text{for} \quad C_L = C_{L_{max}} \quad \text{or} \quad C_{L_{min}} \tag{31}$$

$$\mu_1 = 0 \quad \text{for} \quad C_L = C_{L_{int}} \tag{32}$$

$$\mu_2 \geq 0 \quad \text{for} \quad Tw = Tw_{max} \quad \text{or} \quad Tw_{min} \tag{33}$$

$$\mu_2 = 0 \quad \text{for} \quad Tw_{int} \tag{34}$$

$$\begin{aligned}& \lambda_1 (Tw - Sw\, e^{-\kappa\eta} M^2 C_D - \sin\gamma) + (\lambda_2/M)(Sw\, e^{-\kappa\eta} M^2 C_L \\ & - \cos\gamma) + \lambda_3 M \cos\gamma + \lambda_4 M \sin\gamma = -1\end{aligned} \tag{35}$$

These equations are similar to those for the simpler dynamic model, except that $\lambda_4' \neq 0$, the additional Lagrange multipliers μ_1 and μ_2 enter the λ_1' and λ_4' equations, and the term $e^{-\kappa\eta}$ appears in various equations. Once again it can be shown that intermediate thrust arcs are nonoptimal because of the generalized Legendre-Clebsch condition. Intermediate lift arcs with minimum thrust are nonoptimal, leaving five possible control combinations, as before. The restrictions on the transitions between these subarcs are the same as with the simplified model.

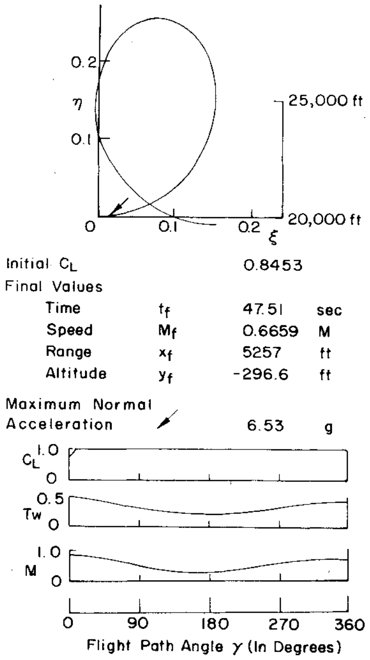

Fig. 6 Control and state histories: $C_{L_{max}} = 1.0$, $(Tw_{max})_i = 0.5$ (more realistic model without n_{max} constraint).

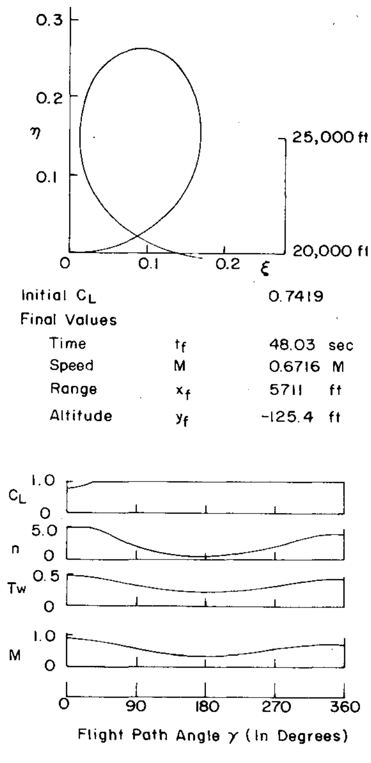

Fig. 7 Control and state histories: $C_{L_{max}} = 1.0$, $n_{max} = 5.0$, $(Tw_{max})_i = 0.5$ (more realistic model with n_{max} constraint).

Numerical Results for the More Realistic Model

Changes in atmospheric pressure with altitude are now considered. C_{D0} and K are assumed to be functions of Mach number, and Tw_{max} and Tw_{min} are assumed to be functions of Mach number and altitude. Constraints on normal acceleration are not considered. An optimal path for the case $C_{L_{max}} = 1.0$, $(Tw_{max})_i = 0.5$ is shown in Fig. 6. The terminal Mach number, range, and altitude are again considered to be free parameters. The control history and the trajectory itself are similar to those shown in Fig. 1 for the simplified model with the same values of $C_{L_{max}}$ and $(Tw_{max})_i$. The time required for the maneuver, the peak altitude, and the final range are somewhat larger than with the simplified model.

If an appropriate normal acceleration constraint is introduced, the trajectory begins with an arc on which C_L is limited by this constraint. This is followed by a $C_{L_{max}}$ arc on which the normal acceleration constraint is ineffective. The trajectory and control history for this case are shown in Fig. 7. The same values of $C_{L_{max}}$ and $(Tw_{max})_i$ as in Fig. 6 are assumed. The trajectory is very similar to that in Fig. 6. The normal acceleration constraint is effective only at the beginning of the maneuver, where the dynamic pressure is largest.

Conclusion

Optimal values of lift coefficient and thrust have been determined as functions of the state variables and Lagrange multipliers for an arbitrary minimum-time maneuver in the vertical plane, using a four-state variable model of the aircraft dynamics. Intermediate-thrust singular arcs have been shown to be nonoptimal by application of the generalized Legendre-Clebsch condition. Intermediate-lift arcs with minimum thrust have also been shown to be nonoptimal. Rules have been established governing the joining together of trajectory segments with different types of control behavior.

For a given set of initial conditions, an extensive study has been made of the control history required for a minimum-time loop maneuver as the maximum thrust and maximum lift coefficient are varied. The final speed, range, and altitude were assumed unspecified. The Tw_{max} vs $C_{L_{max}}$ plane has been divided into 11 regions. Nine of these regions are characterized by qualitatively different control programs for minimum-time loop maneuvers. One of the remaining two has an admissible solution but no stationary solutions. In the other region, the maneuver cannot be completed. When $C_{L_{max}}$ and Tw_{max} are relatively large, the loop is generated mainly by means of lift control rather than thrust. When $C_{L_{max}}$ and Tw_{max} are relatively small, both lift and thrust are important control variables.

The requirement of maximum thrust at a high angle of attack and low Mach number, near the top of the loop maneuver, imposes severe restrictions on the inlet, duct, and engine design from the standpoint of compressor stall of the engine. The results obtained using the more realistic aerodynamic model do not differ appreciably from those obtained with the simplified model.

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